GRAMSOL

NEW CURRICULUM

A-Level Pure Mathematics

Form 6

Approved by Ministry of Primary and Secondary Education

UNITY • FREEDOM • WORK

ZIMBABWE

Stanley Maunze

A - level
Pure Mathematics

Form 6

Stanley Maunze

GRAMSOL

Published by
Gramsol Books
46 Kwame Nkrumah Avenue,
Harare. 2019
www.gramsol.com
Tel. 0864 420 9124

Publisher: Sandura Tafadzwa
Managing Editor: Mbono Njabulo
Commisioning editor: Mabhodyera Armstrong
Development editor: Mbono Kudakwashe
Typeset by Mhlahlo Artkins
Cover by Chikanga Francis

Printed by Gramsol Books

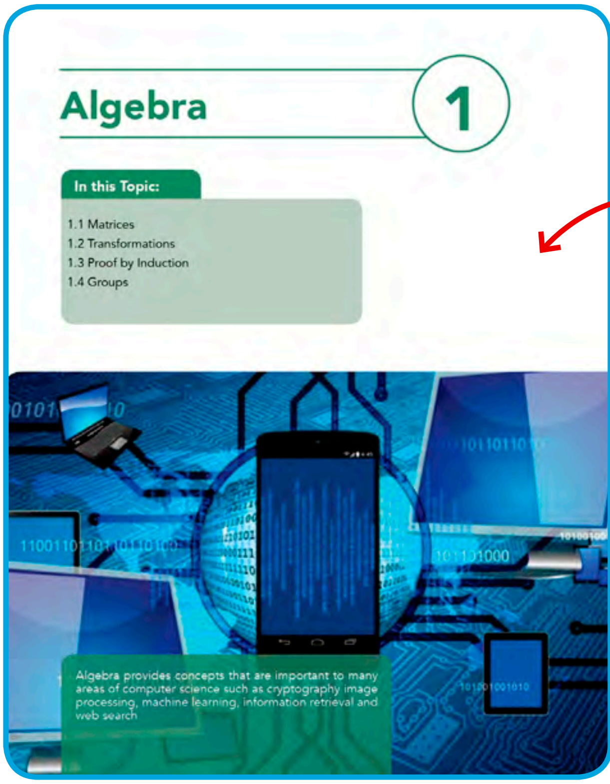
ISBN: 9781779297563

Acknowledgements
The publisher and authors would like to acknowledge and thank the following for photographs and copyright material: Pexels / Pixabay

Although we have tried to trace and contact all copyright holders before publication, this has not been possible in all cases. If notified the publisher will rectify any errors or omissions at the earliest opportunity.

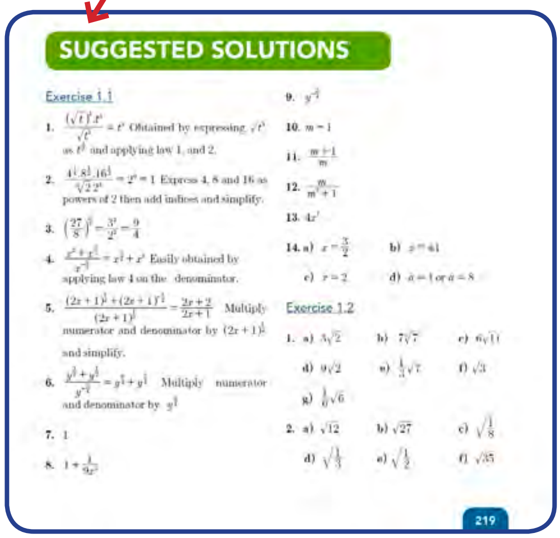
Contents

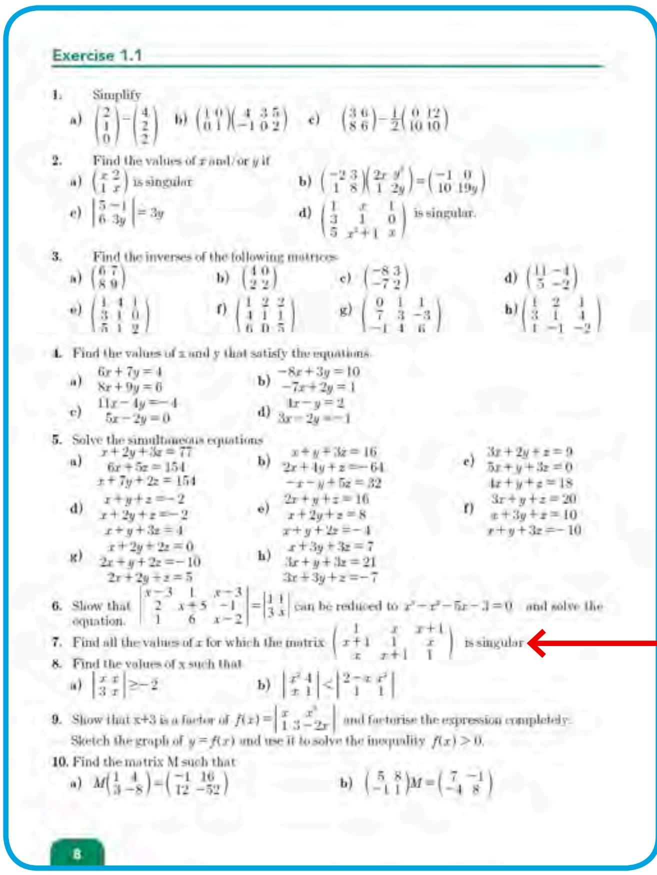
Book Features



Opening page with introduction

Suggested solutions for every exercise and test





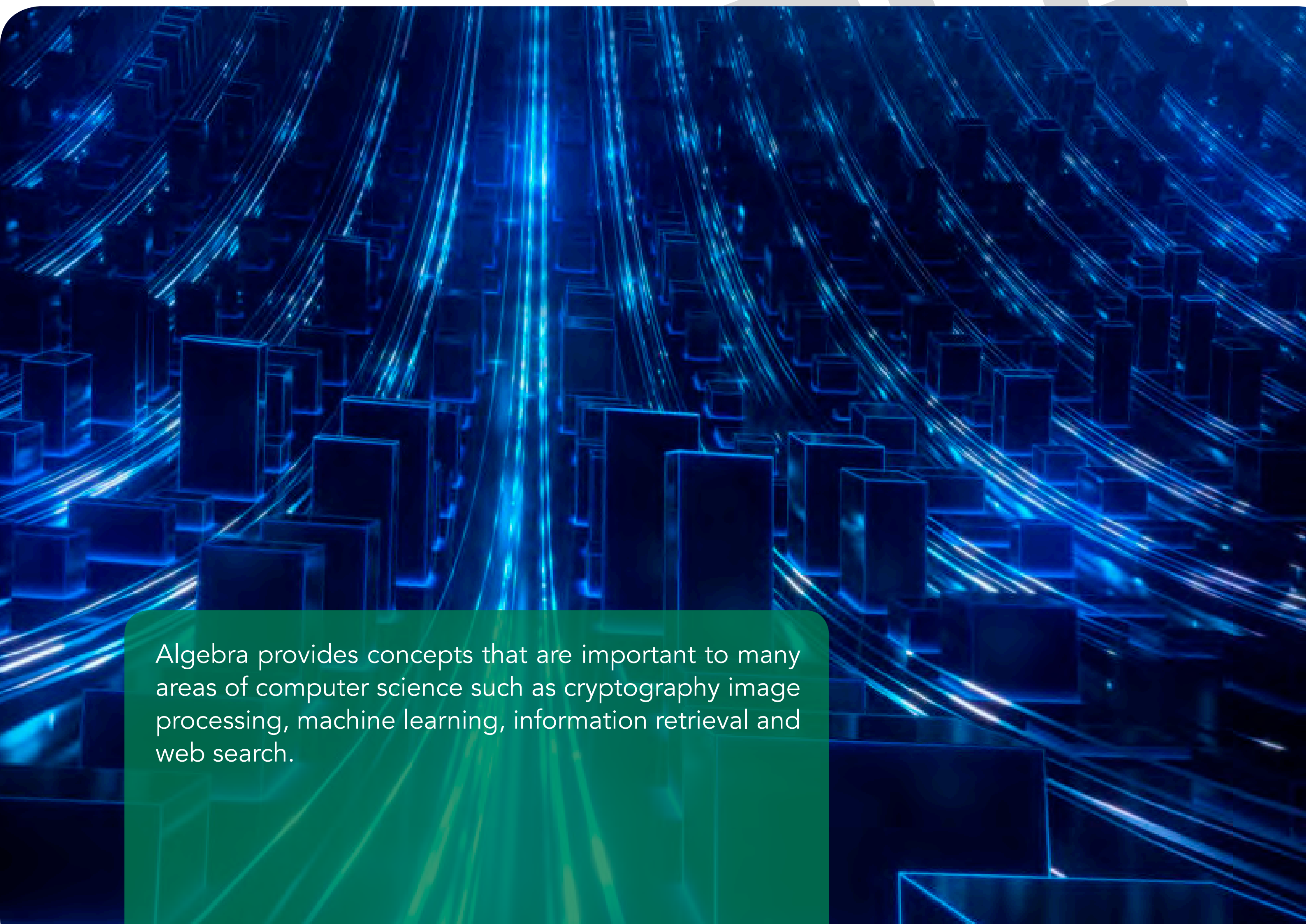
Exercise at the end of each unit

Algebra

1

In this Topic:

1.1 Matrices
1.2 Transformations
1.3 Mathematical Induction
1.4 Groups



Algebra provides concepts that are important to many areas of computer science such as cryptography image processing, machine learning, information retrieval and web search.

1.1 MATRICES

Objectives

By the end of this section, the learner should be able to:

- add, subtract and multiply matrices and recognise the terms null (or zero) matrix and identity matrix (or unit matrix).
- recall the meaning of the terms singular and non-singular as applied to square matrices.
- evaluate determinants and find inverses of non-singular 2×2 or 3×3 matrices.
- use the result $AB)^{-1} = B^{-1}A^{-1}$ for non-singular matrices.
- formulate a problem involving 2 linear simultaneous equations in two unknowns or 3 equations in 3 unknowns, as a problem involving the solution of a matrix equation.

A matrix is an ordered array of numbers in rows and columns. A matrix with equal number of rows and columns is called a square matrix. The order of a matrix is specified as $r \times n$ where r is the number of rows and n is the number of columns.

Addition and subtraction

Addition and subtraction is done by adding or subtracting corresponding entries of the matrices.

Example 1

a) $\begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -4 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2-4 & 0+4 \\ 3+1 & 4+2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 4 & 6 \end{pmatrix}$

b) $2\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 3\begin{pmatrix} 4 \\ 5 \end{pmatrix} += \begin{pmatrix} 2\times 1 \\ -1\times 2 \end{pmatrix} - \begin{pmatrix} 3\times 4 \\ 3\times 5 \end{pmatrix} + \begin{pmatrix} 8 \\ 10 \end{pmatrix} = \begin{pmatrix} 2-12+8 \\ -2-15+10 \end{pmatrix} = \begin{pmatrix} -2 \\ -7 \end{pmatrix}$

c) $\begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ cannot be simplified because the entries do not correspond.

d) $\begin{pmatrix} 4 & 3 & 2 \\ 1 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & -6 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 4-1 & 3-2 & 2-1 \\ 1-3 & -1-2 & 0-(-6) \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ -2 & -3 & 6 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and no further simplification is possible because the matrices that result do not have the same number of rows and columns.

The transpose of a matrix A is written AT and it is the matrix that results when the rows and columns of A are interchanged; row 1 to column 1, row 2 to column 2 and so on.

Example 5

1. $A=\begin{pmatrix}-2&-1&1\\4&0&-5\\1&-1&3\end{pmatrix}\text{adj}(A)=\begin{pmatrix}-5&-17&-4\\2&-7&-3\\5&-6&4\end{pmatrix}^T=\begin{pmatrix}-5&2&5\\-17&-7&-6\\-4&-3&4\end{pmatrix}$

The following shows how the entries of adj**a)** are obtained.

For row 1 of A with entries $-2\ -1\ 1$ gives column 1 of adj(**A**) as follows:

the cofactor of -2 is $+\begin{vmatrix}0&-5\\-1&3\end{vmatrix}=0-5=-5$ which is the first entry of adj(**A**)

the cofactor of -1 is $-\begin{vmatrix}4&-5\\1&3\end{vmatrix}=-(12-(-5))=-17$ and

the cofactor of 1 is $+\begin{vmatrix}4&0\\1&-1\end{vmatrix}=-4-0=-4$

$\Rightarrow$ column 1 has entries $-5\ \ -17\ \ -4$ in adj(**A**) as given above.

For row 2 of A with entries $4\ 0\ -5$ we have column 2 of adj(**A**) as follows:

the cofactor of 4 as $-\begin{vmatrix}-1&1\\-1&3\end{vmatrix}=-(-3+1)=2$

the cofactor of 0 as $+\begin{vmatrix}-2&1\\1&3\end{vmatrix}=-6-1=-7$

the cofactor of -5 as $-\begin{vmatrix}-2&-1\\1&-1\end{vmatrix}=-(2+1)=-3$

$\Rightarrow$ column 2 of adj(**A**) has entries $2\ -7\ -3$

For row 3 of A with entries $1\ -1\ 3$ we get column 3 of adj**a)** as follows:

the cofactor of 1 as $+\begin{vmatrix}-1&1\\0&-5\end{vmatrix}=5-0=5$

the cofactor of -1 as $-\begin{vmatrix}-2&-1\\4&-5\end{vmatrix}=-(10-4)=-6$

the cofactor of 3 as $+\begin{vmatrix}-2&-1\\4&0\end{vmatrix}=0-(-4)=4$

$\Rightarrow$ column 3 of adj(**A**) has entries $5\ -6\ 4$

Hence $A\times adj(A)=\begin{pmatrix}-2&-1&1\\4&0&-5\\1&-1&3\end{pmatrix}\begin{pmatrix}-5&2&5\\-17&-7&-6\\14&-3&4\end{pmatrix}=\begin{pmatrix}23&0&0\\0&23&0\\0&0&23\end{pmatrix}$

$$=23\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=23I \text{ and } \det(A)=23$$

$$A^{-1}=\frac{1}{23}\begin{pmatrix}-5&2&5\\-17&-7&-6\\-4&-3&4\end{pmatrix}$$

2. Given that $B = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ 1 & 2 & 5 \end{pmatrix} \text{adj}(B) = \begin{pmatrix} 5-8 & -(10-4) & 4-1 \\ -(0-6) & 5-3 & -(2-0) \\ 0-3 & -(4-6) & 1-0 \end{pmatrix}^T$

$$= \begin{pmatrix} -3 & -6 & 3 \\ 6 & 2 & -2 \\ -3 & 2 & 1 \end{pmatrix}^T = \begin{pmatrix} -3 & 6 & -3 \\ -6 & 2 & 2 \\ 3 & -2 & 1 \end{pmatrix}$$

and therefore $B \times adj(B) = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} -3 & 6 & -3 \\ -6 & 2 & 2 \\ 3 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 6I$

From which $\det(\mathbf{A}) = 6$ and $B^{-1} = \frac{1}{6} \begin{pmatrix} -3 & 6 & -3 \\ -6 & 2 & 2 \\ 3 & -2 & 1 \end{pmatrix}$

Alternatively one can find the cofactors of BT and then form the adjoin of B.

Simultaneous equations

Any system of simultaneous equations can be expressed in matrix form and conversely any matrix equation can be expressed in terms of simultaneous equations associated with it.

Simultaneous equations with 2 unknowns of the forms $ax + by = m,\ cx + dy = n$ can be expressed in matrix form as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix}$ by taking the coefficients of x and y in their order. Solution of the equations can be achieved by pre-multiplying the matrix equation by the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which is $\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Simultaneous equations in 3 unknowns x,y and z in the forms:

$a_1x + b_1y + c_1z = p,\ a_2x + b_2y + c_2z = q,\ a_3x + b_3y + c_3z = r$

can be written as $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

The matrix equation is solved by pre-multiplying with the inverse of $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ therefore

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\text{adj}(B)\times B=\begin{pmatrix}-5&2&5\\-17&-7&-6\\-4&-3&4\end{pmatrix}\begin{pmatrix}-2&-1&1\\4&0&-5\\1&-1&3\end{pmatrix}=\begin{pmatrix}23&0&0\\0&23&0\\0&0&23\end{pmatrix}=23I$$

$$\text{Det}(B)=23 \text{ and } B^{-1}=\frac{1}{23}\begin{pmatrix}-5&2&5\\-17&-7&-6\\-4&-3&4\end{pmatrix}$$

$$\text{Therefore }\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{23}\begin{pmatrix}-5&2&5\\-17&-7&-6\\-4&-3&4\end{pmatrix}\begin{pmatrix}-46\\23\\23\end{pmatrix}=\begin{pmatrix}-5&2&5\\-17&-7&-6\\-4&-3&4\end{pmatrix}\begin{pmatrix}-2\\1\\1\end{pmatrix}=\begin{pmatrix}17\\21\\9\end{pmatrix}$$

Notice that 23 was factored out before the multiplication.

Note: The three simultaneouys equations can be regarded as cartesian equations of three planes. The solution of the three equations give a point of intersection of three planes. [Refer to topic on Vectors]

Exercise 1.1

1. Simplify.

a) $\begin{pmatrix}2\\1\\0\end{pmatrix}-\begin{pmatrix}4\\2\\2\end{pmatrix}$ b) $\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}4&3&5\\-1&0&2\end{pmatrix}$ c) $\begin{pmatrix}3&6\\8&6\end{pmatrix}-\frac{1}{2}\begin{pmatrix}0&12\\10&10\end{pmatrix}$

2. Find the values of x and/or y if:

a) $\begin{pmatrix}x&2\\1&x\end{pmatrix}$ is singular

b) $\begin{pmatrix}-2&3\\1&8\end{pmatrix}\begin{pmatrix}2x&y^2\\1&2y\end{pmatrix}=\begin{pmatrix}-1&0\\10&19y\end{pmatrix}$

c) $\begin{vmatrix}5&-1\\6&3y\end{vmatrix}=3y$

d) $\begin{pmatrix}1&x&1\\3&1&0\\5&x^2+1&x\end{pmatrix}$ is singular.

3. Find the inverses of the following matrices.

a) $\begin{pmatrix}6&7\\8&9\end{pmatrix}$ b) $\begin{pmatrix}4&0\\2&2\end{pmatrix}$ c) $\begin{pmatrix}-8&3\\-7&2\end{pmatrix}$ d) $\begin{pmatrix}11&-4\\5&-2\end{pmatrix}$

e) $\begin{pmatrix}1&4&1\\3&1&0\\5&1&2\end{pmatrix}$ f) $\begin{pmatrix}1&2&2\\4&1&1\\6&0&5\end{pmatrix}$ g) $\begin{pmatrix}0&1&1\\7&3&-3\\-1&4&6\end{pmatrix}$ h) $\begin{pmatrix}1&2&1\\3&1&4\\1&-1&-2\end{pmatrix}$

4. Find the values of x and y that satisfy the equations.

a) $\begin{aligned}6x+7y&=4\\8x+9y&=6\end{aligned}$

b) $\begin{aligned}-8x+3y&=10\\-7x+2y&=1\end{aligned}$

c) $\begin{aligned}11x-4y&=-4\\5x-2y&=0\end{aligned}$

d) $\begin{aligned}4x-y&=2\\3x-2y&=-1\end{aligned}$

Example 6

Solve the equations

a) $\begin{aligned} x+2y&=0 \\ 3x+4y&=4 \end{aligned}$ b) $\begin{aligned} 2x-y&=7 \\ -5x+3y&=1 \end{aligned}$ c) $\begin{aligned} x+3z&=1 \\ 2x+y+4z&=0 \\ x+2y+5z&=1 \end{aligned}$ d) $\begin{aligned} z-2x-y&=-46 \\ 4x-5z&=23 \\ x-y+3z&=23 \end{aligned}$

a) In matrix form the equations become $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

det $= 4-6=-2$ and the inverse is $\frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}=\frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix}=\frac{1}{-2}\begin{pmatrix} -8 \\ 4 \end{pmatrix}=\begin{pmatrix} 4 \\ -2 \end{pmatrix}$

b) The matrix equation is $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 7 \\ 1 \end{pmatrix}$ and the inverse is $\frac{1}{6-5}\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}=\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \end{pmatrix}=\begin{pmatrix} 22 \\ 37 \end{pmatrix}$

c) The matrix equation is $\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}=\mathrm{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

We can transpose A first: $\mathrm{AT}=\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}$ then find the matrix of cofactors of AT

$\mathrm{adj}(\mathrm{A})=\begin{pmatrix} -3 & 6 & -3 \\ -6 & 2 & 2 \\ 3 & -2 & 1 \end{pmatrix}$

$\mathrm{A}\times\mathrm{adj}(\mathrm{A})\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} -3 & 6 & -3 \\ -6 & 2 & 2 \\ 3 & -2 & 1 \end{pmatrix}=\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

Hence $\det(\mathbf{A})=6$ and $A^{-1}=\frac{1}{6}\begin{pmatrix} -3 & 6 & -3 \\ -6 & 2 & 2 \\ 3 & -2 & 1 \end{pmatrix}$

therefore the solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\frac{1}{6}\begin{pmatrix} -3 & 6 & -3 \\ -6 & 2 & 2 \\ 3 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}=\frac{1}{6}\begin{pmatrix} 0 \\ -8 \\ 2 \end{pmatrix}=\begin{pmatrix} 0 \\ -\frac{4}{3} \\ \frac{1}{3} \end{pmatrix}$

d) The matrix equation is $\begin{pmatrix} -2 & -1 & 1 \\ 4 & 0 & -5 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -46 \\ 23 \\ 23 \end{pmatrix}=\mathrm{B}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\mathrm{B}^{\mathrm{T}}=\begin{pmatrix} -2 & 4 & 1 \\ -1 & 0 & -1 \\ 1 & -5 & 3 \end{pmatrix}\Rightarrow \mathrm{adj}(\mathrm{B})=\begin{pmatrix} -5 & 2 & 5 \\ -17 & -7 & -6 \\ -4 & -3 & 4 \end{pmatrix}$

Geometry and Vectors

2

Vectors are used in Structural Engineering. They are used to calculate the forces and moments that are acting on a structure to make sure that these are not bigger than what the structure can withstand.

Objectives

By the end of this section, the learner should be able to:

- find and use the Cartesian equation of a line in 3D
- find and use cross product of given vectors to find area of plane shapes.
- find and use vector, Cartesian and parametric equations of a plane
- find and use the angle between lines, between line and plane and between two planes.
- determine whether lines are parallel, skew or intersecting,
- find the perpendicular distance of a point from a line or plane, or distance of a line from a plane, or distance between parallel lines or parallel planes.
- determine whether a line lies in, is parallel to, or intersects a plane and find the angle of intersection where necessary
- find the point of intersection of a line and plane
- find the equation of the line of intersection of two planes.

Vector Presentation

There are basically two ways of representing vectors:-

a) The column format $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in which the entries can be two or three; as in the example depending on whether its two or three dimensions.

b) The $\boldsymbol{i}$–$\boldsymbol{j}$–$\boldsymbol{k}$ format. Two dimensional vectors need only $\boldsymbol{i}$ and $\boldsymbol{j}$. While three dimensional vectors use the three. For example $\boldsymbol{v} = 2\boldsymbol{i} + 3\boldsymbol{j} - 8\boldsymbol{k} = \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix}$.

In two dimensions $\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

In three dimensions $\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are **unit vectors** and are called **basis vectors**, that is, vectors in terms of which any other vector can be expressed.

Magnitude of a vector

In 3-Dimensions the magnitude of a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is given by $\sqrt{x^2 + y^2 + z^2}$

In 2-Dimensions the magnitude of a vector $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ is given by $|\boldsymbol{v}| = \sqrt{x^2 + y^2}$

A **unit vector** is any vector with magnitude 1.

The Scalar Product of vectors

The scalar product of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is written $\boldsymbol{v.w}$ and calculated as shown below.

If $\boldsymbol{v} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ and $w\begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix}$ then $\boldsymbol{v.w} = 1\times 5 + (-2)\times 0 + 4\times 6 = 5 - 0 + 24 = 29$

a) When vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are ***perpendicular*** their dot product is zero, $\boldsymbol{v.w} = 0$.
Conversely when their dot product is zero the two vectors are perpendicular.

b) Given that the position vectors of the points A and B are respectively $\boldsymbol{a}$ and $\boldsymbol{b}$ the midpoint M of the line **AB** has position vector $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}) = \mathbf{OM}$

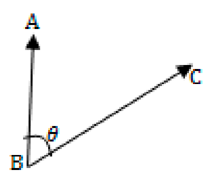
c) The cosine formula for the angle between two vectors $\boldsymbol{a} = \mathbf{AB}$ and $\boldsymbol{b} = \mathbf{CB}$

$\boldsymbol{a.b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ where θ is the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

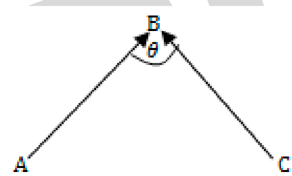
It is very important to note that when considering the directions of vectors **AB** and **CB**:-

directions of both **AB** and **CB** must **converge** to their point of intersection B.

or directions of both **AB** and **CB** must **diverge** from their point of intersection B.



Both diverge from the point B

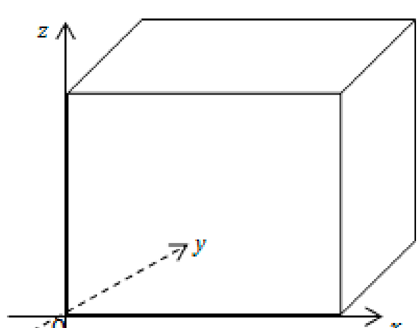


Both converge to the point B

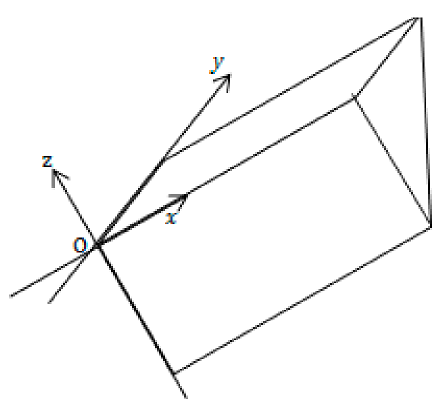
This will ensure the correct angle between AB and CB is obtained instead of $(180° - \theta)$ or its supplement.

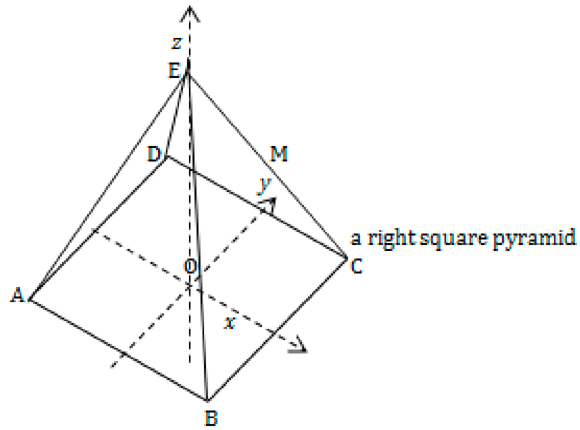
The Cartesian frame of reference

Vectors in 3 dimensions can be represented on basic solid shapes using the sides of the solid shapes like cuboids, cubes, prisms, pyramids, etc. One can set the origin at any of the vertices or even on a side or edge of the solid shape as shown in the following diagrams.



Cuboid





In the cuboid the origin is set at the left hand vertex. In the triangular pyramid, the origin is set at the left hand vertex. In the square pyramid, the origin is set at the centre of the square base.

The angles and sides in any problem involving the shape will be found by considering the distances measured parallel to the given axes as the coordinates. For example, the coordinates of point C on the square base of the square pyramid can be $(2,2,0)$ if the square base has sides 4 units. The position vector of C is thus $\mathbf{OC} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = 2\boldsymbol{i} + 2\boldsymbol{j}$. If the height of the pyramid is 6 units the position vector of E will be $\mathbf{OE} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$, and M, the midpoint of CE, will have position vector

$\mathbf{OM} = \frac{1}{2}\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$.

A point with position vector $\begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix}$ will be 3 units vertically below the centre of the base of the pyramid.

Line Vectors

The equations of a line

The equation of a line can be expressed in vector form, Cartesian form or parametric form.

The vector equation of a line passing through two points with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written as $\boldsymbol{r} = \boldsymbol{a} + \lambda(\boldsymbol{b} - \boldsymbol{a})$. The line is parallel to the direction of $\boldsymbol{b} - \boldsymbol{a}$.

The vector equation of a line through a point with position vector $\boldsymbol{a}$ and parallel to a direction of vector $\boldsymbol{m}$ is written $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{m}$.

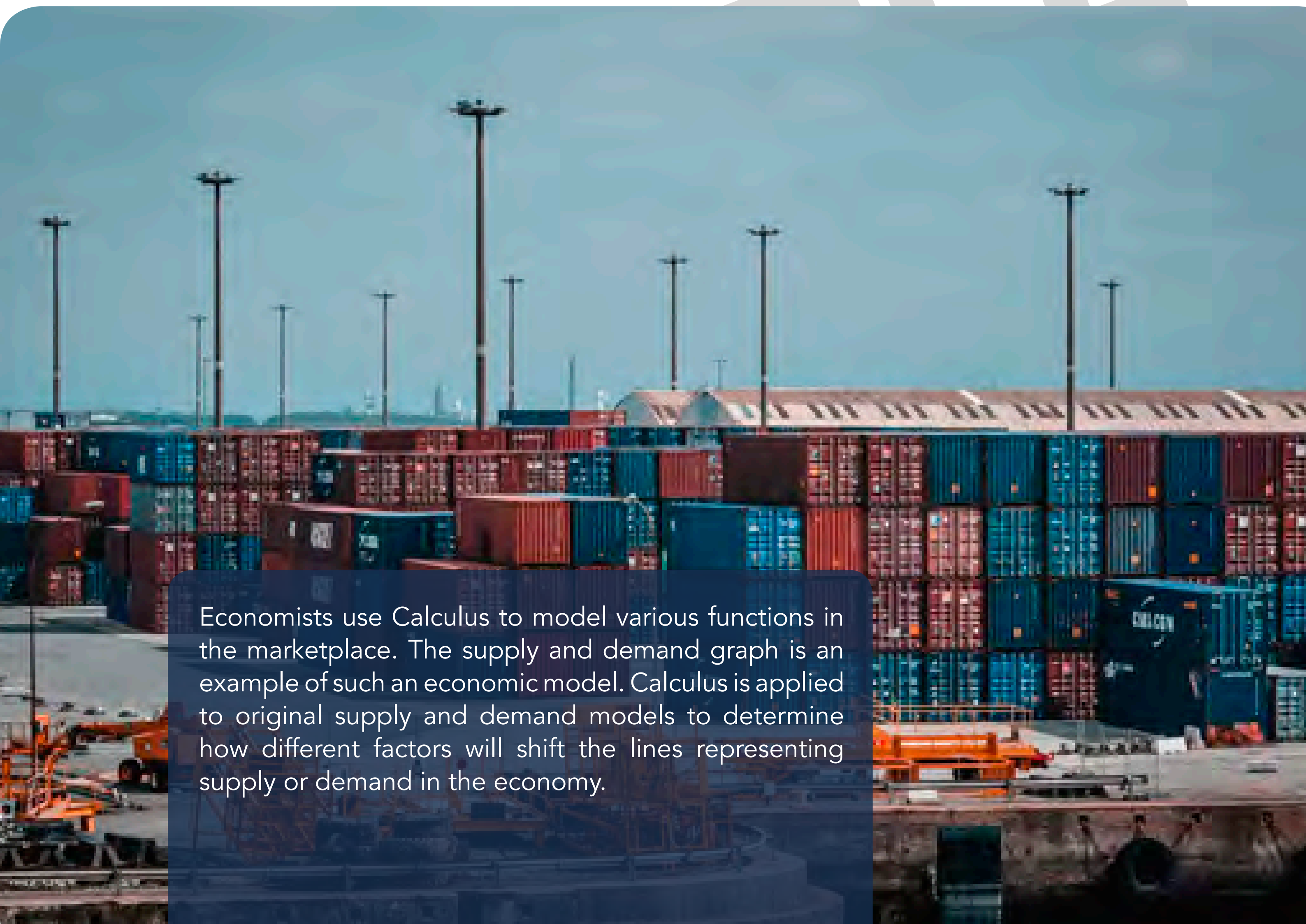
The parametric forms of the equation of a line are:

$x = x_1 + a \quad y = y_1 + b \quad z = z_1 + c$

The Cartesian form of the equation of a line is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$.

A line parallel to the vector $b = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and passing through the point $A(2, -1, 3)$ has equation

Calculus: Ordinary Differential Equations 4



Economists use Calculus to model various functions in the marketplace. The supply and demand graph is an example of such an economic model. Calculus is applied to original supply and demand models to determine how different factors will shift the lines representing supply or demand in the economy.

Objectives

By the end of this section the learner should be able to:

- formulate simple statements involving a rate of change as a differential equation, including the introduction of a constant of proportionality,
- find by integration a general form of solution for a differential equation in which the variables are separable,
- represent the general solution of a differential equation by means of a graphical sketch,
- use an initial condition to find a particular solution of the differential equation.

A differential equation is an equation in which two or more variables are connected in a relationship involving a derivative. In this chapter, we will restrict to equations involving the first derivative only.

Solving First Order Differential Equations

There are two types of problems to be solved by separating variables according to syllabus requirements:

a) those given as algebraic expressions involving derivatives for example; $x^2\frac{dy}{dx} = y+1$.

b) those given in words and require formulation of the algebraic expressions.

The learners are expected to formulate differential equations from given statements concerning rates of change. A decreasing rate of change is usually indicated by a negative sign. Rates of change are given as proportional to some variable in the problem being solved. The reader needs to recall the concept of proportionality.

Separating variables requires that the variables in the equation be algebraically manipulated to a form in which each side of the equation has only one variable. The variables x and y in the example $x^2\frac{dy}{dx} = y+1$ can be separated by dividing both sides by x^2 and by $y+1$ and then integrating both sides of the resulting equation $\frac{1}{y+1}\frac{dy}{dx} = \frac{1}{x}^{2}$ with respect to x as follows: $\int \frac{1}{y+1}\frac{dy}{dx}dx = \int \frac{1}{x}^{2}dx$. The right−hand side differential operator $\frac{dy}{dx}dx$ will algebraically simplify to dy and one can then integrate the function in y on the right hand side with respect to y.

$$\int \frac{1}{y+1}dy = \int \frac{1}{x}^{2}dx \Rightarrow \ln(y+1) = -\frac{1}{x}+c$$

The integral $\ln(y+1) = -\frac{1}{x}+c$ is a general solution of the differential equation.

Initial conditions or other conditions on the variables will be given in most differential equations and these must be substituted in the general solution to obtain a particular solution of the differential equation where a numerical value of the constant c is found.

Example 1

Solve the differential equations

a) $\frac{dy}{dx}=\frac{x^2}{y}$ **b)** $x\frac{dy}{dx}=1-y^2$ **c)** $\frac{dy}{dx}=\frac{1-x}{y}$

a) $\frac{dy}{dx}=\frac{x^2}{y}$ The variables are separated by cross-multiplying y and integrating with respect to x:

$$y\frac{dy}{dx}=x^2 \Rightarrow \int y dy=\int x^2 dx$$

$$\Rightarrow \frac{y^2}{2}=\frac{x^3}{3}+c$$

b) $x\frac{dy}{dx}=1-y^2$

Divide by x and divide by $1-y^2$ to separate variables:

$\frac{1}{1-y^2}\frac{dy}{dx}=\frac{1}{x}$ which factorises to $\frac{1}{(1-y)(1+y)}\frac{dy}{dx}=\frac{1}{x}$

Express $\frac{1}{(1-y)(1+y)}$ in partial fractions using cover-up method:

$$\frac{1}{(1-y)(1+y)}=\frac{1}{(1-y)(1+1)}+\frac{1}{1-(-1)(1+y)}=\frac{1}{2}\left(\frac{1}{1-y}+\frac{1}{1+y}\right)$$

The differential equation becomes $\frac{1}{2}\left(\frac{1}{1-y}+\frac{1}{1+y}\right)\frac{dy}{dx}=\frac{1}{x}$

Integrating with respect to x we have

$$\frac{1}{2}\left[\int\left(\frac{1}{1-y}+\frac{1}{1+y}\right)dy\right]=\int\frac{1}{x}dx$$

$$\Rightarrow \frac{1}{2}[-\ln(1-y)+\ln(1+y)]=\ln x+\ln c$$

Applying laws of logarithms we have:

$$\Rightarrow \frac{1}{2}\left[\ln\left(\frac{1+y}{1-y}\right)\right]=\ln cx \Rightarrow \ln\left(\frac{1+y}{1-y}\right)=2\ln cx=\ln(cx)^2\equiv\ln(ax^2)$$

where the constant a replaces constant c^2

Exponentiating gives $e^{\ln\left(\frac{1+y}{1-y}\right)} = e^{\ln(ax^2)}$

and the solution simplifies to $\dfrac{1+y}{1-y} = ax^2$

Making y the subject of the formula we express y in terms of x:

$1 + y = (1-y)ax^2 = ax^2 - yax^2 \Rightarrow y + yax^2 = ax^2 - 1$

$\Rightarrow y(1+ax^2) = ax^2 - 1$

The solution is $y = \dfrac{ax^2 - 1}{1 + ax^2}$

c) $\dfrac{dy}{dx} = \dfrac{1-x}{y}$ is solved by separating variables and integrating.

Variables are separated by multiplication by y so that there is no term in y on the right hand side.

$y\dfrac{dy}{dx} = \dfrac{1-x}{y}y$ becomes $y\dfrac{dy}{dx} = 1 - x$

Integrating with respect to x we have;

$\int y\dfrac{dy}{dx}dx = \int(1-x)dx \Rightarrow \int ydy = \int(1-x)dx$

$\dfrac{y^2}{2} = x - \dfrac{x^2}{2} + c$ is the general solution of the equation.

Example 2

Traders who use the Whatsapp social media for advertising their wares claim that the rate of spread of their advertisements is proportional to the sum of the number of people aware of the advertisement, x, and the size of the Whatsapp group, g, at the time of first advertising by a member of the group.

a) Express x, g and time t in a differential equation.

b) Solve the differential equation in the case where g people are aware of an advertisement at time $t = 0$. For what values of the constant of proportionality is the solution valid?

c) Sketch the graph of x against t.

a) The proportionality statement says that $\dfrac{dx}{dt} \propto x + g$

The differential equation is $\dfrac{dx}{dt} = k(x+g)$ where k is the constant of proportionality.

The size of the group, g, is also a constant.

b) Variables are separated by dividing by $x+g$:

$$\frac{1}{x+g}\frac{dx}{dt}=k \Rightarrow \int \frac{1}{x+g}\frac{dx}{dt}dt=\int kdt \Rightarrow \ln(x+g)=kt+c$$

Exponentiation yields $e^{\ln(x+g)}=e^{kt+c}$

$\Rightarrow x+g=e^{kt+c}=e^{k}te^{c}=Ae^{k}t$ where $A=e^{c}$ after applying the first law of indices.

Substituting the initial conditions: $x=g$, $t=0$ we have $g+g=Ae^{k}(0)$

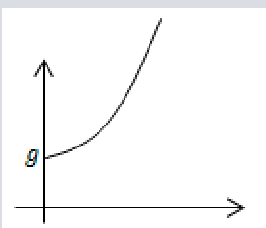
$\Rightarrow 2g=Ae^{0}=A(1)=A$

$\Rightarrow x+g=2ge^{kt}$

$\Rightarrow$ the particular solution is $x=2ge^{kt}-g$

The solution is valid for positive values of k since the case describes an increase in the number of people aware of an advertisement.

c)



The graph indicates exponential growth in awareness of the advertisement.

Example 3

Maize is being lost due to infestation by weevils from a poorly constructed storage tank. The weekly reduction in mass of maize is proportional to the factor $m-5$ where m, measured in tonnes, is the mass of maize at the end of t weeks.

a) Form a differential equation connecting m, t and a constant.

b) Solve the differential equation expressing m in terms of t given that the tank was initially filled with 100 tonnes of maize.

c) Sketch a graph of mass against time for this situation.

a) The weekly reduction is actually a rate of reduction in mass and $\frac{dm}{dt} \propto m-5$

The differential equation is $\frac{dm}{dt}=-k(m-5)$

b) The negative constant indicates a reduction in the mass of maize. The reader may note that the solution can still be established without the negative sign on k by choosing the value of k less than zero, that is, $k < 0$, which represents a reduction in value.

Separating variables we have: $\frac{1}{m-5}\frac{dm}{dt} = -k$

Integrating with respect to t we have: $\int \frac{1}{m-5} dm = \int -k dt \Rightarrow \ln(m-5) = -kt + c$

Exponentiating gives : $e^{\ln(m-5)} = e^{-kt+c} \Rightarrow m - 5 = e^{-kt+c} = e^{-kt}e^{c}$

$\Rightarrow m - 5 = Ae^{-kt}$ where the constant A replaces the constant e^{c}

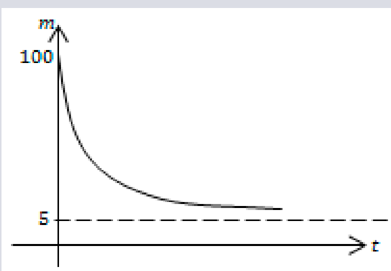
Initially, that is, at time $t = 0$, the tank had $m = 100$ tonnes and substituting these initial conditions into the general solution gives

$100 - 5 = Ae^{-k(0)} = Ae^{0} = A(1) = A \Rightarrow A = 95$

The particular solution is $m = 5 + 95e^{-kt}$

c) The graph of $m = 5 + 95e^{-kt}$ shown below indicates that mass starts at 100 and decays exponentially with time.

This is called exponential decay.



The graph has an asymptote at $m = 5$ because $5 + 95e^{-kt}$ approaches $5 + 0$ when t approaches infinity, that is, $m = 5$ at infinity. The minimum mass is 5 tonnes.

Exercise 4.1

Solve the differential equations

1. $e^{-t}\frac{ds}{dt} = \sqrt{s}$ given that $s = \frac{1}{4}$ when $t = 0$. Sketch the graph of s against t.
2. $e^{2x-y}\frac{dy}{dx} = 2$, given that $y = 0$ when $x = 0$.

3. $x\tan y\dfrac{dy}{dx}=x+1$

4. $\dfrac{dy}{dx}=\dfrac{2-y}{2-x}$. Express y in terms of x. Describe the graph of y against x.

5. $\dfrac{dy}{dx}=\dfrac{y+1}{x^2-1}$ given that the curve passes through the point $(2;0)$

6. $(x^2+1)\dfrac{dy}{dx}=2xy^3$ given that $x=0$ when $y=1$

7. $e^y\dfrac{dy}{dx}=x^4e^{ky}$ given that $y=0$ when $x=0$.

8. $\dfrac{dx}{d\theta}=x\cos^2 2\theta$

9. Given that $\dfrac{dy}{dx}=(\tan x)\sqrt{y}$ for $0\le x<\dfrac{\pi}{2}$ and $y=1$ when $x=0$, find an expression for y in terms of x.

10. One kilogram of stewed beef is placed in a cooler box. It is assumed that the rate of decrease in temperature of the beef is proportional to θ, where $\theta°C$ is the temperature of the stewed beef at time t minutes after being placed in the cooler box.

When $t=0$, $\theta=80°C$ and after 3 minutes $\theta=40°C$.

a) Form a differential equation relating θ and t.

b) Solve the differential equation and find the temperature of the stewed beef after a further 3 minutes.

c) Sketch the graph of temperature against time for this situation.

11. Solve the differential equation $(4-x)\dfrac{dy}{dx}=y$, given that $y=4$ when $x=1$, expressing y in terms of x.

12. The rate of spread of falsehoods after their first introduction into the social media is jointly proportional to the number of individuals who have received the falsehood x, and the number of individuals who are yet to receive the falsehoods, $N-x$, where N is the number of subscribers on the social network.

a) Express this statement as a differential equation relating x and the time t in the case where a network has 1000 subscribers.

b) Given that one person introduced the falsehoods at $t=0$, express x in terms of t.

13. Solve the differential equation $e^{-2t}\dfrac{ds}{dt}=s^2$ given that $s=-2$ when $t=0$ expressing s in terms of t.

Given that t is time find the time it takes for s to reduce to half the initial value leaving your answer in exact form.

14. Water is being pumped into a leaking tank and the tank is filling up at a rate of $v-5$ cubic metres per hour. Initially the tank had $6m^3$ of water. The pump is switched off after pumping

for only 5 hours and the water now leaks at the rate of $v-5$ cubic metres per hour.

Form differential equations for the situation and solve them. Find the volume of water in the tank after a further 5 hours. What is the minimum volume of water in the tank?

Sketch, on the same axes, the graphs of volume against time during and after pumping. Your graphs must cover at least 10 hours.

15. After administering gibberellic acid to unsprouted potato tubers an agronomist notices that the number of tubers that sprouted was decreasing at a rate proportional to n per hour, where n is the number of treated tubers that were yet to sprout. The agronomist had treated 500 tubers and he found out that 250 tubers had not yet sprouted 5 hours after treatment.

Form a differential equation for this situation and solve it. Sketch a graph of n, the number of tubers yet to sprout, against time t.

16. In a chemical reaction in which a compound P is formed from a compound Q, the masses of P and Q present at time t are x and y respectively. The sum of the masses x and y is 10 and at any time the rate at which x is increasing is proportional to the product of the two masses at that time.

a) Show that $\frac{dx}{dt}=kx(10-x)$ where k is a constant.

b) Find the general solution of the differential equation.

Hence, find t correct to 3 significant figures when $x=9.9$ given that $x=2$ when $t=0$ and $x=5$ when $t=\ln 2$.

17. A girl returning from a milling point is carrying mealie-meal in a cylindrical container. The container has a hole at the bottom and the mealie-meal trickles out through this hole. It is estimated that the rate of reduction of mealie-meal is proportional to the mass m of mealie-meal remaining in the container, so that the situation is modeled by the differential equation $\frac{dm}{dt}=-\frac{k}{5}m$ where k is a constant.

Find the general solution of the differential equation and show that it reduces to $m=m_o e^{-\frac{k}{5}t}$ where m_o is the initial mass of the mealie-meal.

The girl takes two hours to walk from the milling point to her home. Given that after one hour, ten percent of the mealie-meal is lost,

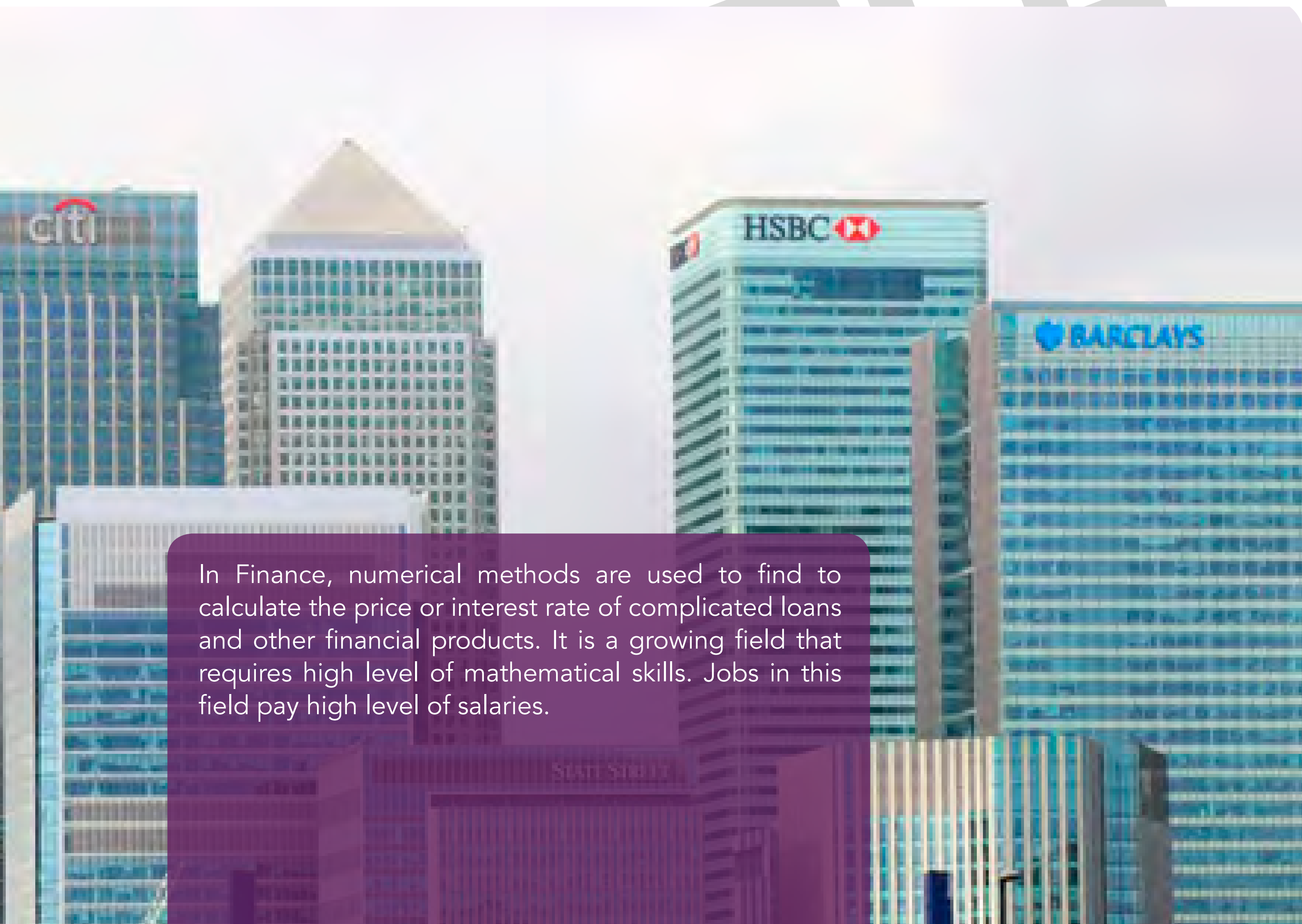
a) calculate the percentage of mealie-meal in the container when she arrives at home.

b) sketch a graph showing the variation of the mass of the mealie-meal during the two-hour journey.

Important terms used in the chapter

exponential decay, infinity, exponential growth, exponentiating, initial conditions, general solution, particular solution, integration, constant of proportionality, differential equation, rate of change

Numerical Methods 5



In Finance, numerical methods are used to find to calculate the price or interest rate of complicated loans and other financial products. It is a growing field that requires high level of mathematical skills. Jobs in this field pay high level of salaries.

Objectives

By the end of this section the learner should be able to:

- distinguish between absolute and relative errors in data,
- make estimates of the errors that can arise in calculations involving inexact data including the use of $\delta y \cong \frac{dy}{dx}\delta x$,
- locate approximately a root of an equation by graphical means or sign change,
- use the idea of a sequence of approximations which converge to the root,
- understand how a given iterative formula of the form $x_{n+1} = F(x_n)$ relates to the equation being solved and use a given iteration to find a root of the equation,
- understand in geometric terms the working of the Newton-Raphson method and derive and use iterations based on this method, understand that an iterative method may fail to converge to a required root,
- use the trapezium rule to estimate area under a curve or set bounds for the area.

Errors

Absolute error is the actual difference between the expected or true value and the estimated value together with the units of the variable where possible.

Relative error is a fraction or percentage formed by expressing the absolute error as a fraction of the expected or true value.

Small Changes

The result $\delta y \cong \frac{dy}{dx}\delta x$ is used for estimating small changes in x and/or y.

The trigonometric approximations below are used when x is small or when x is near zero.

$$\sin x \cong x \qquad \cos x \cong 1 - \frac{x^2}{2} \qquad \tan x \cong x$$

Example 1

a) Given that $y = \sqrt{x}$ find the percentage change in y when x increases by 0.5%.

Use the first derivative: $\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}} = \frac{1}{2}\cdot\frac{\sqrt{x}}{x} = \frac{1}{2}\cdot\frac{y}{x}$ by algebraic manipulation.

Substitute this derivative into the result $\delta y \cong \frac{dy}{dx}\delta x$ and obtain:

$\delta y \cong \frac{1}{2}.\frac{y}{x}\delta x = \frac{1}{2}.\frac{\delta x}{x}.y\frac{\delta y}{y}\times 100\% = \frac{1}{2}.\frac{\delta x}{x}\times 100\%$ by dividing both sides by y.

Then $\frac{\delta y}{y}\times 100\% = \frac{1}{2}\times 0.5\% = 0.25\%$ increase in y.

b) The variables x and y are related by the equation $y = \frac{10}{1-x}$.

Use the first derivative of y with respect to x to find the change in x that results when y changes from 5 to 5.02

When $y = 5$ we have $5 = \frac{10}{1-x} \Rightarrow x = 1 - \frac{10}{5} =- 1$.

The small change in y is 0.02

$y = \frac{10}{1-x} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(10(1-x)^{-1}) = 10(-1)(-1)(1-x)^{-2} = \frac{10}{(1-x)^2}$

Substituting the derivative into the result $\delta y \cong \frac{dy}{dx}\delta x$ yields:

$0.021 \cong \frac{0}{(1-(-1))^2}.\delta x$

$\Rightarrow \delta x = \frac{0.02(1-(-1))^2}{10} = 0.008$ is the increase in x.

c) Given that x is small show that

i) $\frac{1+\sin\left(\frac{\pi}{2}+x\right)}{\cos\left(\frac{\pi}{2}+x\right)} \cong \frac{x}{2} - \frac{2}{x}$

Expanding the numerator we have:

$$1+\sin\left(\frac{\pi}{2}+x\right) = 1 + \sin\frac{\pi}{2}\cos x + \sin x \cos\frac{\pi}{2}$$
$$= 1 + \cos x + 0 = 1 + \cos x$$
$$\cong 1 + 1 - \frac{x^2}{2} = 2 - \frac{x^2}{2}$$

Expanding the denominator we have:

$\cos\left(\frac{\pi}{2}+x\right) = \cos\frac{\pi}{2}\cos x - \sin\frac{\pi}{2}\sin x =- \sin x$

$\cong - x$ when x is small.

The fraction now becomes $\frac{2-\frac{x^2}{2}}{-x} = \frac{x}{2} - \frac{2}{x}$

ii) $(\cos^2(\pi - x))(\sin^2(\pi - x) \cong x^2$ when terms in x^3 and higher are neglected.

Expand $\cos^2(\pi - x)$ and $\sin^2(\pi - x)$ first then square the results:

$$\cos(\pi - x) = \cos\pi\cos x + \sin\pi\sin x = -\cos x$$

$$\cong -\left(1 - \frac{x^2}{2}\right) = \frac{x^2}{2} - 1$$

$$\sin(\pi - x) = \sin\pi\cos x - \sin x\cos\pi = \sin x \cong x$$

Squaring we have: $\cos^2(\pi - x) \cong \left(\frac{x^2}{2} - 1\right)^2 = \frac{x^4}{4} - x^2 + 1$ and $\sin^2(\pi - x) \cong x^2$

Combining the two squares we have:

$$\cos^2(\pi - x)\sin^2(\pi - x) \cong x^2\left(\frac{x^4}{4} - x^2 + 1\right) = \frac{x^6}{4} - x^4 + x^2$$

$$\cong x^2$$

iii) $\tan\left(\frac{\pi}{3} + x\right) \cong \frac{\sqrt{3} + x}{1 - \sqrt{3}x}$

Expand using the multiple angle formula for tan(A+**B**):

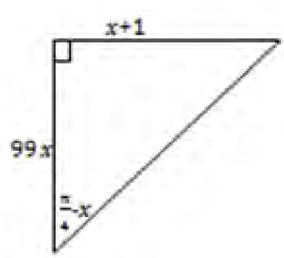
$$\tan\left(\frac{\pi}{3} + x\right) = \frac{\tan\frac{\pi}{3} + \tan x}{1 - \tan\frac{\pi}{3}\tan x} = \frac{\sqrt{3} + x}{1 - \sqrt{3}x}$$

Exercise 5.1

1. Given that x is small such that powers of x higher than 3 can be neglected, simplify as far as possible:

a) $\dfrac{\sin\left(\frac{\pi}{2} + x\right) - \frac{1}{2}}{1 + \cos\left(\frac{\pi}{2} + x\right)}$ (b $\sin\left(\frac{\pi}{4} + x\right)\cos\left(\frac{\pi}{4} + x\right)$ **c)** $\sin\left(\frac{\pi}{4} + x\right)\cos\left(\frac{\pi}{4} - x\right)$

2. Given that x is small enough for x^3 and higher powers to be neglected find the approximate value of x in the right-angled triangle below in which the angle is given as $\frac{\pi}{4} - x$.



3. Given that x is small enough for x^3 and higher powers to be neglected express the area of the triangle below as a polynomial in x.

Values of x are: 1 1.4 1.8 2.2 2.6 3.0 by adding 0.4 successively.

Value of y are: $\frac{2}{\sqrt{3}}$ $\frac{2.8}{\sqrt{3.4}}$ $\frac{3.6}{\sqrt{3.8}}$ $\frac{4.4}{\sqrt{4.2}}$ $\frac{5.2}{\sqrt{4.6}}$ $\frac{6}{\sqrt{5}}$

Substituting these values of y in the trapezium formula:

$$\int_1^3 \frac{2x}{\sqrt{x+2}}dx \cong \frac{0.4}{2}\left[\frac{2}{\sqrt{3}}+\frac{6}{\sqrt{5}}+2\left(\frac{2.8}{\sqrt{3.4}}+\frac{3.6}{\sqrt{3.8}}+\frac{4.4}{\sqrt{4.2}}+\frac{5.2}{\sqrt{4.6}}\right)\right]$$

$\cong 2.972$

Exercise 5.3

1. Use the trapezium rule with three equal intervals to estimate $\int_0^{0.6} xe^x\,dx$

Evaluate the integral exactly and hence calculate the percentage error in your estimate.

2. Use the trapezium rule to estimate the integrals as indicated.

a) $\int_0^2 \sqrt{x^2+1}\,dx$ by dividing the area into four equal intervals.

b) $\int_0^4 \frac{x^2}{1+x}dx$ using 5 ordinates. Evaluate the exact integral and find the percentage error in the estimate. Does the trapezium rule over–state or under–state the true area under the curve?

c) $\int_1^9 \log_{10} x\,dx$ using 8 equal intervals.

d) $\int_{\frac{\pi}{12}}^{\frac{\pi}{3}} \cot x\,dx$ using 4 ordinates. Evaluate the integral and find the percentage error in the trapezium rule estimate. Deduce whether the trapezium rule over–states or under–states the area under the curve.

3. Use the trapezium rule with 4 equal intervals to estimate the value of $\int_0^{0,4} \frac{\sin x^2}{e^{x^2}}dx$ giving your answer correct to 4 decimal places.

4. Use the trapezium rule with three equal intervals to estimate $\int_2^8 \frac{x}{\sqrt{1+x^2}}dx$ correct to 2 decimal places.

Differentiate $\sqrt{1+x^2}$ and hence evaluate exactly $\int_2^8 \frac{x}{\sqrt{1+x^2}}dx$ and find the percentage error in your estimate.

5. Use the trapezium rule with 5 ordinates to obtain an approximate value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{\sin\theta}\,d\theta$

6. Use the trapezium rule with 4 equal intervals to estimate the value of $\int_0^{0.2} \cos\frac{x^2}{x^2}dx$ giving your answer correct to 4 decimal places.

7. a) Use the trapezium rule with 5 ordinates to evaluate

$\int_0^1 \frac{4}{1+x^2}dx$ correct to 4 decimal places.

b) i) By using the substitution $x = \tan\theta$ find $\int_0^1 \frac{4}{1+x^2}dx$

(ii) hence find correct to 2 decimal places the percentage error in using the trapezium rule as an approximation to the integral.

8. a) Use the trapezium rule with 4 equal intervals to estimate the value of

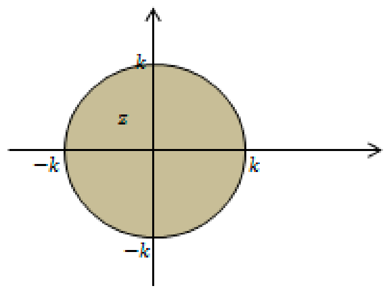
$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \text{cosec}^2 x dx$ dx giving your answer correct to 3 decimal places.

b) Evaluate exactly $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \text{cosec}^2 x dx$

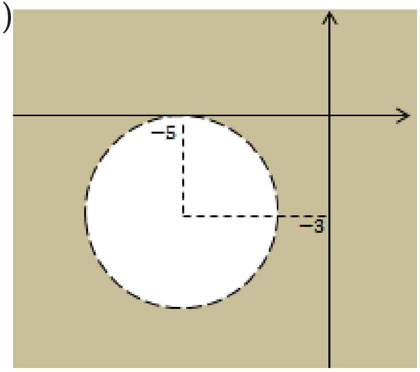
c) Calculate the percentage error in using the trapezium rule as an approximation to the integral.

Important terms used in the chapter

trapezium rule, number of intervals, ordinates, overstating, understating, change of sign, Newton-Raphson Method, small changes, relative error, absolute error, trapezium rule, converge, iterative formula, sign change, root of an equation c)

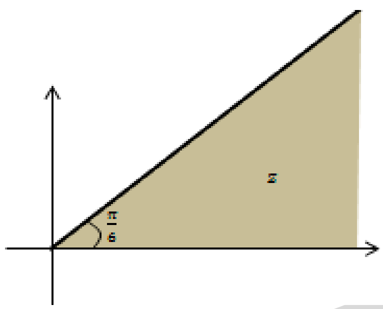


d)

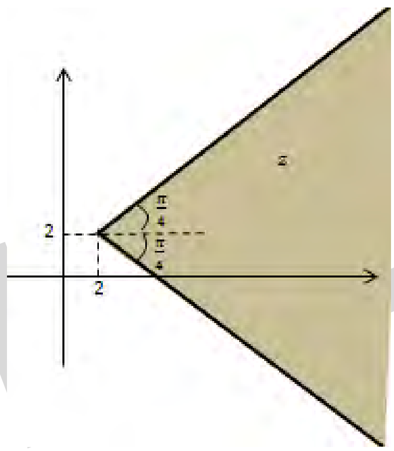


7. Describe the locus represented in each of the following Argand diagrams and state the locus in algebraic form.

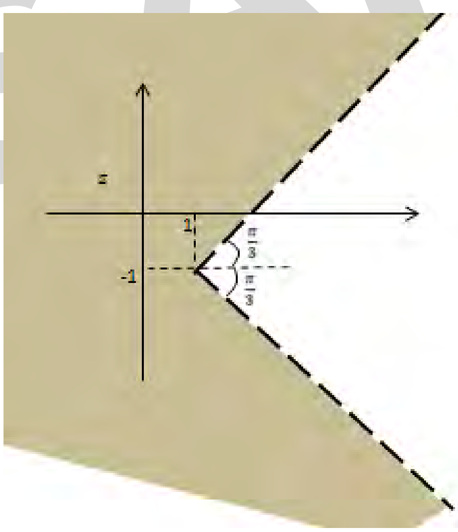
a)



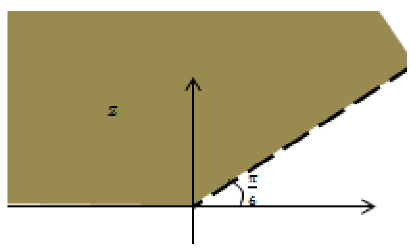
b)



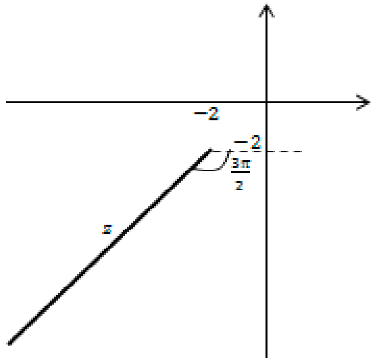
c)



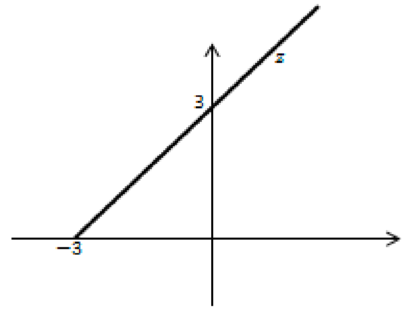
d)



e)

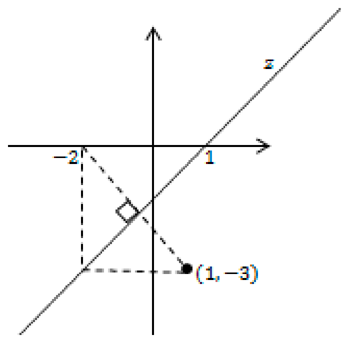


f)

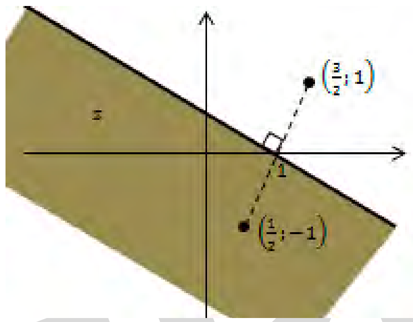


8. Describe the locus represented by the Argand diagram. Give your answer in the form of an equation or inequality. Use the given points.

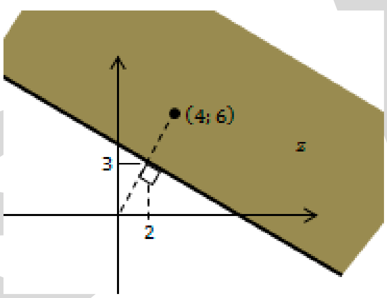
a)



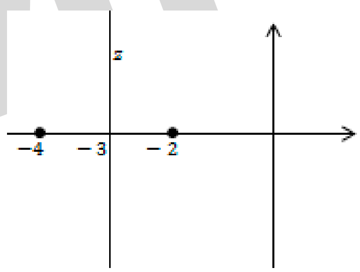
b)



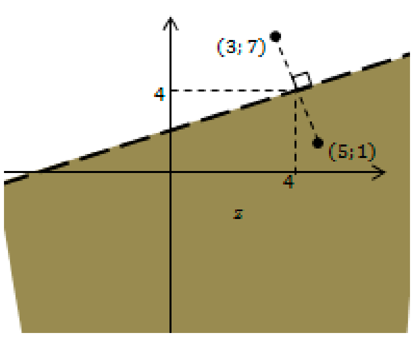
c)



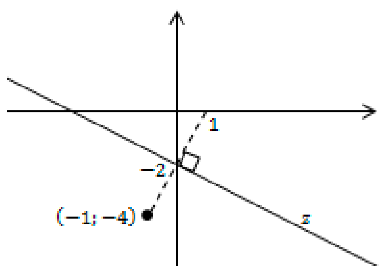
d)



e)



f)



Examination Practice Papers

Paper 1A

1. Solve the equation $4^x + 4^{x+2} = 34$. [3]

2. Find the term independent of x in the expansion of $\left(\frac{x^3}{2} + \frac{2}{x}\right)^8$ [3]

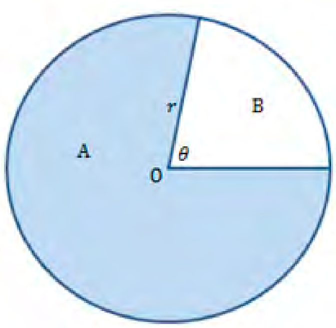
3. Given that $y = \frac{3}{\sqrt[3]{x}}$ use differentiation to estimate the percentage change in y when x is reduced by 6%. [4]

4. The complex number $w = \frac{1-2i}{2+i}$.

a) Express w in the form $x + iy$ where x and y are both real numbers and hence sketch the complex number w in an Argand diagram. [3]

b) Given that $\frac{3+i}{a+bi} = 1 + i$ find the real number a and b. [2]

5.



The diagram above shows a circle of radius r centred at O. The circle is divided into two sectors A and B in such a way that the perimeter of A is twice the perimeter of B. The angle at the centre of sector B is θ radians.

a) Show that $\theta = \frac{2\pi - 2}{3}$. [3]

b) Hence express the ratio of area of B to the area of A exactly in terms of π. [3]

6. a) Express $\cos\theta + 2\sin\theta$ in the form $R\sin(\theta + \alpha)$ where $R > 0$ and α is acute. [2]

b) Hence, or otherwise, solve the equation $\cos\theta + 2\sin\theta = 0$ for $0^\circ \le \theta \le 360^\circ$. [3]

c) Hence sketch the graph of $y = \frac{1}{\cos\theta + 2\sin\theta}$ for values of θ up to 360°. [2]

7. The position vectors of the points A, B and C are given by

$\overrightarrow{OA} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\overrightarrow{OB} = 2\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$, $\overrightarrow{OC} = -\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$.

Find

a) angle $A\hat{B}C$. [4]

b) the exact distance from A to C. [2]

What can you say about triangle ABC? [1]

8. The functions f and g are defined by

$f: x \to \ln x, x > 0$ and $g: x \to \dfrac{1}{x+1}$, $x \neq -1$

a) Find $fg(x)$ in its simplest form and state its domain. [3]

b) Describe the geometrical transformations that map $f(x)$ onto $fg(x)$. [3]

c) Find the inverse of $fg(x)$ and state its domain. [3]

9. a) Express $\dfrac{x^2-1}{x(x^2+1)}$ as a sum of two fractions. [5]

b) Hence find the exact value of $\displaystyle\int_1^2 \frac{x^2-1}{x(x^2+1)}dx$. [4]

10. A curve has equation $y = 4x^2 + \dfrac{1}{x}$.

a) Find the value of $\dfrac{dy}{dx}$ at the instant when $x = 1$. [3]

b) The rate of change of y with time is 0.7 units per second.

Find the rate of change of x at the instant when $x = 1$. [2]

c) Find the exact area enclosed between the curve, the x-axis and the lines $x = 1$ and $x = 2$. [4]

11. a) Find the first term and common ratio of a geometric progression in which the second term is 1 and the sum to infinity is 4. [5]

b) In an arithmetic progression of common difference 1 the sum of the first n terms is −55 and the sum of the first $2n$ terms is −10.

Find n and the first term. [6]

INDEX

S

T

U

V